Covering Points with Disjoint Unit Disks

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Abstract

We consider the following problem. How many points must be placed in the plane so that no collection of disjoint unit disks can cover them? The answer, k, is already known to satisfy $11 \leq k \leq 53$. Here, we improve the lower bound to 13 and the upper bound to 50. We also provide a set of 45 points that apparently cannot be covered, although this has been determined via computer search.

1 Introduction

In 2008, Japanese puzzle designer Naoki Inaba proposed and solved an interesting question [3, 4], which was to determine if every given configuration of 10 points can be covered by identical coins. Any number of coins can be used, but they cannot overlap. That is, Inaba proved the following lower bound.

Theorem 1 [Inaba] *Any configuration of 10 points in the plane can be covered by disjoint unit disks.*

Inaba gave an interesting proof based on the probabilistic method (see [6, 8]), and asked the natural extension: *How many points do we need to use, so that their appropriate arrangement cannot be covered by disjoint unit disks?*

Let k be the size of the smallest point set that is not coverable. Inaba's theorem states that $11 \leq k$, and trivially k is finite; if we place sufficiently many points on a fine lattice, disjoint disks cannot cover them all (see Figure 1). This problem gained popularity within the puzzle society in 2010 (at the 9th *Gathering 4 Gardner*). Winkler [7] proposed a configuration of 60 points that cannot be covered by disjoint disks. Winkler also suggested how to improve the lower bound in [8], but this has not been settled[1]. Elser [1] improved the upper bound to 55, and Okayama et. al [6] further improved this to 53.

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[1]Personal communication with Peter Winkler.

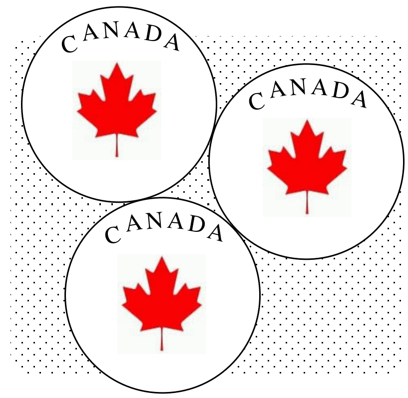

Figure 1: A dense point set that cannot be covered by disjoint unit disks.

In this paper, we improve the known bounds as follows.

Theorem 2 *Let k be the size of the smallest point set that is not coverable by disjoint unit disks. Then $13 \leq k \leq 45$.*

That is, we improve the lower bound from 11 to 13, and the upper bound from 53 to 45.

For the lower bound, we give a refinement of Inaba's proof based on the probabilistic method. For the upper bound, we have used two different approaches. First we give a configuration of 50 points on a lattice, for which an analytical proof exists. This is an improvement over the solution in [6], from which we remove three points. We also state a better upper bound of 45. The validity of this configuration has been determined via an exhaustive computer search, however we note that a mathematically rigorous proof remains to be shown. Finally we mention that it is NP-complete to decide if a given set of n points can be covered [2].

2 Preliminaries

We say that a disk C is *placed at* (x, y) if its center is placed at the point (x, y). This is sometimes denoted by $C(x, y)$. To simplify our arguments, let each unit disk be open, so that it does not cover points on its boundary. Using a perturbation technique, our results can be applied to closed disks as well. We denote by $|A|$ the area of a bounded region A in the plane. A

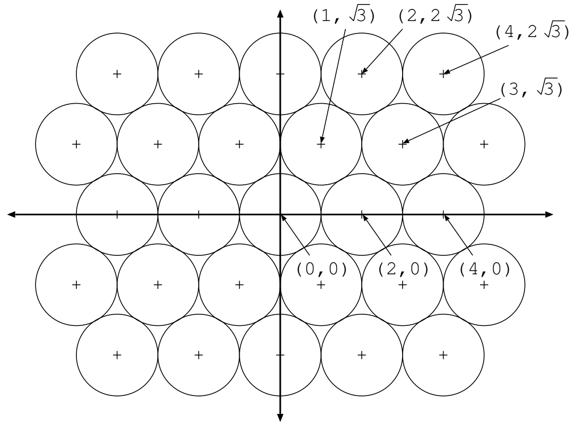

Figure 2: An infinite configuration $S(0,0)$ of unit disks.

region S in the plane is *periodic* if there is a bounded region $A \subset \mathbb{R}^2$ such that for any vector $(x, y) \in \mathbb{R}^2$ there is a vector $(a, b) \in A$ with $S + (x, y) = S + (a, b)$. A measurable minimum-area set A with this property is then said to be a *fundamental region* for S. The *density* $\rho(S)$ of S is defined by

$$\rho(S) = \frac{|S \cap A|}{|A|}$$

which is independent of the choice of A, if A is a fundamental region for S.

3 Lower bound

In this section, we show that $13 \leq k$. That is, any set of 12 points can be covered by disjoint unit disks. Let S be the union of unit disks placed at $(2i + (j \bmod 2), j\sqrt{3})$, for $i, j \in \mathbb{Z}$. (See Figure 2.) S is then periodic. Its fundamental set is the regular hexagon H with vertices at $(\pm 1, \pm\sqrt{3}/3)$ and $(0, \pm 2\sqrt{3}/3)$. Its density is thus $\rho(S) = |S \cap H|/|H| = \pi/\sqrt{12} \sim 0.9069$. We will show that any 12 points in the plane are covered by $S + (x, y)$ for some $(x, y) \in H$.

Inaba's proof for 10 points notes that if (x, y) is chosen uniformly from H then any fixed point in the plane is covered by $S + (x, y)$ with probability $\rho(S) > 9/10$. Thus, of any 10 points, the expected number covered by $S + (x, y)$ is greater than 9; it follows that with positive probability $S + (x, y)$ will cover all ten points. Of course, only at most 10 of the disks in $S + (x, y)$ are actually needed to cover the points.

We refine Inaba's method to show:

Lemma 3 *Any configuration of 12 points can be covered by $S + (x, y)$, for appropriate values of x and y.*

Proof. (Outline) We denote by $\overline{S + (x, y)}$ the set of points that are not covered by $S + (x, y)$. Since $(a, b) \in S + (x, y)$ if and only if $(x, y) \in S - (a, b)$, we have:

Observation 1 *Let X be a set of m points $p_1 = (x_1, y_1), p_2 = (x_2, y_2), \ldots, p_m = (x_m, y_m)$. Then the following statements are equivalent.*

1. *For all $(x, y) \in H$, $S + (x, y)$ fails to cover X.*
2. *$\cup_{i=1,\ldots,m} \overline{S - (x_i, y_i)}$ covers the plane;*
3. *$\cup_{i=1,\ldots,m} \overline{S - (x_i, y_i)}$ covers H.*

For $x \in [-1, 1]$, let $P(x)$ be the vertical line segment consisting of all points in H of x-coordinate x. Then the ratio $\phi(x) = \left|\overline{S + (x, y)} \cap P(x)\right| / |P(x)|$ (where the set-measure is now ordinary one-dimensional length) is given by

$$\phi(x) = \left(\sqrt{3} - \sqrt{1 - x^2} - \sqrt{2|x| - x^2}\right) / \sqrt{3}.$$

Thus, by Observation 1, to prove Lemma 3 it is sufficient to establish that

$$\min\{\phi(x) + \phi(x - d_1) + \phi(x - d_2) + \cdots + \phi(x - d_{11})\} < 1$$

for any real numbers $d_1, d_2, \ldots, d_{11}$. Let $\psi(x; d_1, d_2, \ldots, d_{11}) = \phi(x) + \phi(x - d_1) + \phi(x - d_2) + \cdots + \phi(x - d_{11})$. We will prove that $\max\min\{\psi(x; d_1, \ldots, d_{11}) \mid 0 \leq x \leq 1\}$ is given when $d_i = i/12$ for $i = 1, \ldots, 11$, and then $\max\min\{\psi(x; d_1, \ldots, d_{11}) \mid 0 \leq x \leq 1\} \approx 0.942809 < 1$. We omit the details of this calculation, which will be given in the electronic proceedings. □

4 Upper bounds

In this section, we state two upper bounds, by providing configurations of point sets that cannot be covered. Our first set is simply a subset of 50 points taken from the pattern in [6]. In fact this configuration is constrained to a triangular lattice, which permits a concise proof. The second set contains only 45 points and was checked by a computer program that is based on a non-trivial exhaustive search. Although disk placement is not a finite process, we explain how this problem can be solved in a discrete way.

4.1 50-point configuration on a triangular lattice

The configuration is given in Figure 3. This is based on a triangular lattice; the smallest equilateral triangle is on a circle of radius $2\sqrt{3}/3 - 1$. Unless mentioned otherwise, when we use the term "triangle", we will be referring to three points that are mutual neighbors on the lattice, i.e., forming the equilateral triangle mentioned. The radius $2\sqrt{3}/3 - 1$ was chosen to be the largest value satisfying the following property:

Lemma 4 ([6]) *The three points forming a triangle cannot be covered by three disjoint unit disks.*

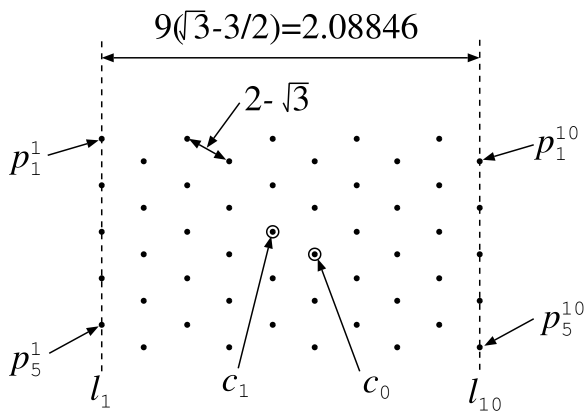

Figure 3: A 50-point configuration that cannot be covered by disjoint unit disks.

Thus, if our set of 50 points is to be covered by disjoint unit disks, each triangle must be covered by either one or two disks. We say that a disk *partially covers* a triangle if it only covers one or two of its points. A *chain* is a sequence of triangles $t_1, t_2, \ldots, t_h$, with consecutive triangles sharing a common edge.

Lemma 5 *Suppose that the chain $t_1, t_2, \ldots, t_h$ is covered, and every disk involved only covers triangles partially. Let C and C' be two disks that cover t_1. Then the entire chain is covered only by C and C'.*

Proof. By Lemma 4, without loss of generality assume that two points p_1, p_2 of t_1 are covered by C, and the remaining point p_3 is covered by C'. Consider the first two triangles ($h = 2$). By definition, t_1 and t_2 share two points. It is not possible for t_2 to share p_1 and p_2, since C cannot cover both points without covering either p_3 or the third point of t_2, which would contradict the assumption about partial coverage by every disk. Thus we can assume that the two triangles share p_1 and p_3, which as mentioned are covered by C and C' respectively. By Lemma 4, the third point of t_2 must be covered by C or C'. Our claim follows by iterating through adjacent pairs of triangles in the chain. Every such pair must share two points that are not covered by the same disk. □

Now we are ready to show the upper bound:

Theorem 6 *The 50-point configuration in Figure 3 cannot be covered by disjoint unit disks.*

Proof. In the configuration, there are 10 vertical columns, each containing 5 points. The columns are labeled ℓ_1 to ℓ_{10} from left to right. Let $p_1^j, p_2^j, p_3^j, p_4^j, p_5^j$ be the five points on ℓ_j from top to bottom. Notice that by rotating a half-turn, the same configuration is obtained.

For the sake of contradiction, suppose that the configuration is covered. Let the *centers* of the configuration be the points $c_0 = p_3^6$ and $c_1 = p_3^5$, as shown in Figure 3. Let C_0 be a unit disk that covers c_0 or c_1. (Choose arbitrarily, if the two centers are covered by two different disks.) It is easy to see that C_0 cannot cover points both in ℓ_1 and in ℓ_{10} since the distance between ℓ_1 and ℓ_{10} is around 2.08846. Without loss of generality, assume that no point in ℓ_1 is covered by C_0. More precisely, suppose that the first r columns are not covered by C_0. Then we have $1 \leq r \leq 5$.

Suppose that p_1^{r+1} and p_5^{r+1} are not covered by C_0. In other words, C_0 covers all points $p_{k_1}^{r+1}, \ldots, p_{k_2}^{r+1}$ for some $2 \leq k_1 \leq k_2 \leq 4$. Then, by Lemma 5, $p_{k_1-1}^{r+1}$ and $p_{k_2+1}^{r+1}$ must be covered by one disk, since a suitable chain connecting the two always exists. However, by convexity, this is impossible.

Therefore, C_0 covers p_1^{r+1} or p_5^{r+1}. We first consider the case that C_0 covers p_5^{r+1}. We distinguish between two subcases, depending on the parity of r as shown in Figure 4 or Figure 5.

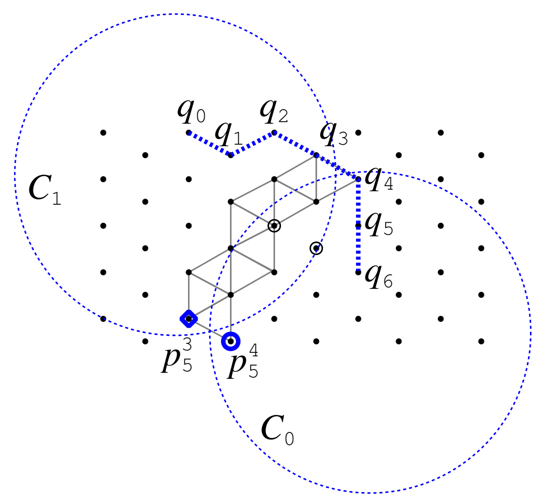

Figure 4: Contradiction for $r = 3$.

Subcase $r = 1, 3, 5$**:** We consider a polyline $L = (q_0, q_1, q_2, q_3, q_4, q_5, q_6)$ defined by $q_0 = p_1^r$, $q_1 = p_1^{r+1}$, $q_2 = p_1^{r+2}$, $q_3 = p_1^{r+3}$, $q_4 = p_2^{r+4}$, $q_5 = p_3^{r+4}$, and $q_6 = p_4^{r+4}$. Figure 4 illustrates the case for $r = 3$. Recall that C_0 covers p_5^{r+1} and at least one of the two centers c_0 and c_1, and does not cover any point in ℓ_r. Given these restrictions, we claim that the boundary of C_0 must cross L. It suffices to show that some vertex of L is contained in C_0, since q_0 is not. For $r = 1$, q_6 lies on the segment joining p_5^{r+1} and c_0. Also, $q_5 = c_1$. Therefore regardless of which center is in C_0, a vertex of L is also in C_0. For $r = 3$ and $r = 5$, C_0 must cover even more of L. Specifically, C_0 cannot reach to cover c_0 or c_1 while containing p_5^{r+1} and excluding q_6.

Let z be the smallest index such that q_z is contained in

C_0. Given the convexity of C_0 as well, we can determine that there must exist a chain of triangles such that every triangle is partially covered by C_0. Furthermore this chain starts with a triangle that has p_5^r, p_5^{r+1} as an edge, and ends with a triangle that has q_{z-1}, q_z as an edge. By Lemma 5, q_{z-1} is covered by the same disk as p_5^r.

An example of a chain is illustrated in Figure 4. Each triangle contains either one or two points covered by C_0. For any given placement of C_0, the chain is easy to determine. Regardless of the value of z, the coverage requirement is geometrically impossible. For example, in Figure 4, to achieve the smallest overlap, C_1 passes through p_5^3 and q_3, and C_0 passes through p_5^4 and q_4 (precisely, C_0 and C_1 are closer since they are open disks). In the case, the distance between the centers of C_0 and C_1 is $\sqrt{3.57198} = 1.88997 < 2$. (Letting $p_5^3 = (0,0), p_5^4 = (2-\sqrt{3}, 0), q_3 = ((2-\sqrt{3})/2, 5(\sqrt{3}-3/2))$, and $q_4 = (3(2-\sqrt{3})/2, 5(\sqrt{3}-3/2))$, we solve $x_1^2+y_1^2 = 1$ and $(x_1 - ((2-\sqrt{3})/2))^2 + (y_1 - (5(\sqrt{3}-3/2)))^2 = 1$ for $C_1(x_1, y_1)$, and $(x_0 - (2-\sqrt{3}))^2 + y_0^2 = 1$ and $(x_0 - (3(2-\sqrt{3})/2))^2 + (y_0 - (5(\sqrt{3}-3/2)))^2 = 1$ for $C_0(x_0, y_0)$. Then we have $(x_0, y_0, x_1, y_1) = (1.14135, 0.487011, -0.739421, 0.673243)$ and $(x_1 - x_0)^2 + (y_1 - y_0)^2 = 3.57198$.) Therefore, C_0 and C_1 overlap in this case. All cases are summarized in Table 1. In each case, the distance is less than 2, and hence the disks C_0 and C_1 overlap.

Case	$q_0 \in C_1$, $q_1 \in C_0$	$q_1 \in C_1$, $q_2 \in C_0$	$q_2 \in C_1$, $q_3 \in C_0$
Distance	1.92528	1.89161	1.88996
Case	$q_3 \in C_1$, $q_4 \in C_0$	$q_4 \in C_1$, $q_5 \in C_0$	$q_5 \in C_1$, $q_6 \in C_0$
Distance	1.88997	1.89160	1.88588

Table 1: Distance in each case (r:odd)

Subcase $r = 2, 4$: In this case, we just change the definition of the polyline $L = (q_0, q_1, q_2, q_3, q_4, q_5)$ as shown in Figure 5 (for $r = 2$). The distances between the two disk centers are summarized in Table 2. In each case, the distance is less than 2. Thus the disks C_0 and C_1 overlap.

Case	$q_0 \in C_1$, $q_1 \in C_0$	$q_1 \in C_1$, $q_2 \in C_0$	$q_2 \in C_1$, $q_3 \in C_0$
Distance	1.92528	1.89161	1.90467
Case	$q_3 \in C_1$, $q_4 \in C_0$	$q_4 \in C_1$, $q_5 \in C_0$	
Distance	1.86166	1.81971	

Table 2: Distance in each case (r:even)

The last case is that C_0 covers p_1^{r+1} (We also know

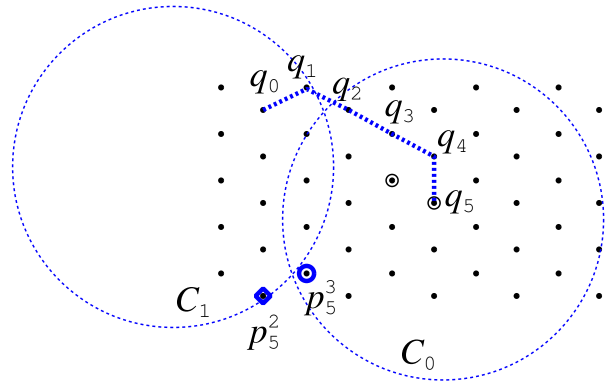

Figure 5: Contradiction for $r = 2$.

that it does not cover p_5^{r+1}, although this does not affect our analysis.) In this case we flip L as shown in Figure 6. We also change how we handle the parity of r, since p_1^{r+1} is on the convex hull if and only if p_5^{r+1} is not.

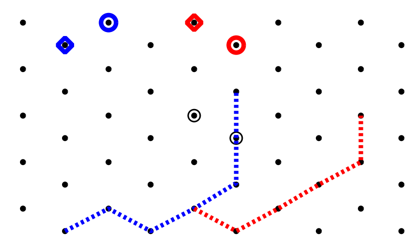

Figure 6: Polylines for $r = 2$ and $r = 5$.

We conclude that in all cases it is impossible to cover all points in Figure 3 with disjoint unit disks. □

4.2 45-point configuration

The configuration given in Figure 7 consists of 45 points equally spaced on three concentric circles: 3 points on the circle of radius 0.1, 21 points on the circle of radius 0.721, and 21 points on the circle of radius 1.0001. By computer search, we have determined that this set can-

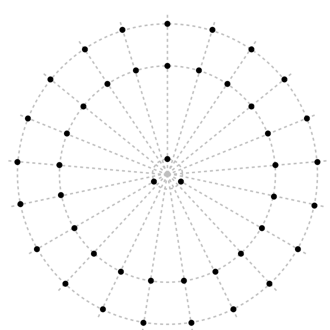

Figure 7: A 45-point configuration that cannot be covered by disjoint unit disks.

not be covered. This set was found heuristically, using the search program interactively.[2]

The search program exhaustively considers all possible ways of covering the given points with disks. This is not obviously a discrete combinatorial search problem — there are an infinite number of possible disk placements. Therefore, we describe here how the search algorithm works.

The algorithm considers (in principle) all possible ways of assigning points to disks that must cover them. This is a discrete search. For each such partition of the points, the algorithm proves conservative bounds for the location of the center of each disk, represented as a rectangle that must contain the disk center. For example, when a disk C is assigned its first point, its rectangle is set to a square of size 2 centered on the point. As more points are assigned to C, the rectangle can be shrunk: no rectangle edge can be farther than 1 from any point p that must be covered. Similar rectangle restrictions may be applied based on the requirement that the disks do not overlap. Sometimes a rectangle may be shrunk to nonexistence, ruling out the current point assignment. This pruning makes the search over all point partitions tractable; most partitions are ruled out without ever being considered explicitly.

If a point assignment survives this first stage of analysis, we are left with a "candidate solution": an assignment of points to disks, and for each disk, a corresponding rectangle. The problem then is to find a solution within this space, or prove that none exists.[3] To do this, we subdivide the largest rectangle, and recursively consider each candidate solution. Eventually, all rectangles are shrunk to the point where either a solution is easy to find (by testing, for example, the rectangle centers), or we can prove impossibility, via the same geometric rectangle restrictions used in the initial part of the search. For example, if we have two disk rectangles that can together be contained in a circle of radius < 2, we can rule out this candidate, because any disk placement respecting these bounds will have overlapping disks (see Figure 8).

Finally, we must mention numerical issues. Our program uses IEEE double-precision floating point numbers. We must ensure that roundoff problems do not cause us to miss a solution. The program uses an adjustable numerical tolerance ϵ for all of its geometrical restrictions — all operations are performed conservatively to this tolerance. (For example, if two points are $< 2+\epsilon$ apart, the program will not rule out the possibility of coverage by a single disk.) This means that in principle, the program could be unable to either find a

[2]This general family of configurations was suggested by Bram Cohen.

[3]At this stage, the problem could also be treated as a quadratically constrained quadratic program, for which solvers exist (e.g., [5]). Our solution is optimized for this particular application.

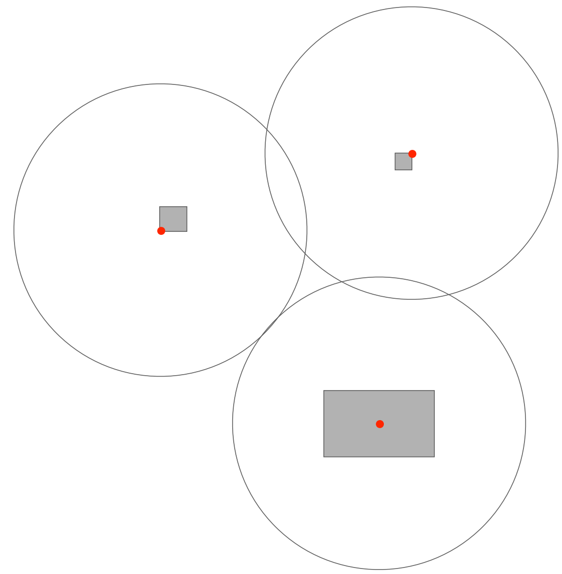

Figure 8: A candidate solution that can be ruled out: any placement will have overlapping disks.

solution or prove that none exists. However, this has not been a problem for the configurations we have searched. IEEE floating point is accurate to 15 decimal places, and we have set $\epsilon = 10^{-5}$, giving us a high confidence that our results are correct.

5 Concluding remarks

We provide lower and upper bounds for the size k of the smallest point set that cannot be covered by disjoint unit disks. Our conclusion is that $13 \le k \le 45$. We conjecture that the true value lies closer to 45. For the lower bound, we have restricted to considering only the fixed configuration in Figure 2 and its translation. By considering rotations and other arrangements of unit disks, the bound might be improved. Moreover, since the bound is (essentially) obtained via the probabilistic method, it is not likely to be tight. We are currently working on a concise mathematical proof for the configuration of 45 points in Figure 7. Small perturbations are not likely to yield improvements. For instance, if the second radius is reduced to 0.720 from 0.721, our program finds a covering. Also, our program has determined that removing any points from the 50-point configuration always yields a covering.

Acknowledgments

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