Edge Guards for Polyhedra in Three-Space

Javier Cano* Csaba D. Tóth† Jorge Urrutia‡

Abstract

It is shown that every polyhedron in $\mathbb{R}^3$ with m edges can be guarded with at most $\frac{27}{32}m$ edge guards. The bound improves to $\frac{5}{6}m + \frac{1}{12}$ if the 1-skeleton of the polyhedron is connected. These are the first non-trivial upper bounds for the edge guard problem for general polyhedra in $\mathbb{R}^3$.

1 Introduction

A *polyhedron* P in $\mathbb{R}^3$ is a compact set bounded by a piecewise linear manifold. Two points, a and b, are *visible* in a polyhedron P if the closed line segment ab is contained in P. For the edges of a polyhedron P, we adapt the notion of *weak visibility*: an edge e of P is visible from a point p if there is a point $q \in e$ such that p and q are visible in P. A set S of edges jointly *guard* P if every point $a \in P$ is visible from some edge in S. It is possible that a point $a \in P$ does not see any vertex of P [11], however, it is not difficult to show that every point $a \in P$ sees at least six edges of P. It follows that every polyhedron with m edges can be guarded by at most $m - 5$ edges.

It was conjectured [14] that any polyhedron of genus zero with m edges can be guarded with at most $\frac{m}{6}$ edge guards. This bound would be optimal apart from an additive constant: for every $k \in \mathbb{N}$, there are polyhedra P_k in $\mathbb{R}^3$ with $6(k+1)$ edges that require at least k edge guards [14], see Figure 1. The polyhedron P_k is the union of a flat tetrahedron T and k pairwise disjoint small tetrahedra attached to one facet of T such that their interiors cannot be seen from any of the edges of T. Since each small tetrahedron has to be guarded by one of its edges, P requires k edge guards.

In this paper, we prove that every polyhedron with m edges (and arbitrary genus) in $\mathbb{R}^3$ can be guarded by at most cm edges, where $c > 0$ is a constant strictly smaller than 1. This is the first nontrivial upper bound for the edge guard problem for general polyhedra. For every polyhedron P in $\mathbb{R}^3$, we choose a set of edges that jointly guard P as the union of two sets: (1) a set of edges that cover all vertices of P, and (2) at most 3/4 of the remaining edges.

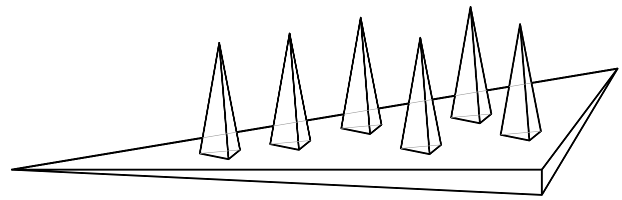


Figure 1: A polyhedron with m edges that requires $m/6 - 1$ edge guards.

The *1-skeleton* of a polyhedron P is the graph defined by the vertices and edges of P. An *edge cover* of a graph $G = (V, E)$ is a set of edges $E_1 \subseteq E$ such that every vertex $v \in V$ is incident to an edge in E_1. By placing guards at every edge in an edge cover of the 1-skeleton of P, we ensure that every point in P that sees a vertex is guarded. Note that the 1-skeleton of P is not necessarily connected (see Figure 1), even if P has genus zero. However, every connected component of the 1-skeleton is 3-connected. In Section 2, using classical matching theory, we give upper bounds for the size of a minimal edge cover in a 3-connected graph, and in a graph formed by the disjoint union of 3-connected components.

In Section 3, we 4-color the edges of P, and show that if a point $a \in P$ does not see any vertex of P, then it sees two edges of different colors. It follows that an edge cover $E_1 \subset E$ and the three smallest color classes of $E \setminus E_1$ jointly guard the entire polyhedron P.

Related work. Most of the previous research on art gallery problems focused on polygons in the plane. For example, it is well known that every simple polygon with n vertices can be guarded by at most $\lfloor \frac{n}{3} \rfloor$ point guards [3], and that every orthogonal polygon with n vertices can be guarded by $\lfloor \frac{n}{4} \rfloor$ point guards [7]. It is widely believed that every simple polygon with n vertices can be guarded by at most $\lfloor \frac{n+1}{4} \rfloor$ of its edges [10].

Everett and Rivera-Campo [6] showed that every triangulated polyhedral terrain in $\mathbb{R}^3$ with n vertices can be guarded by $\lfloor \frac{n}{3} \rfloor$ edges, as $\lfloor \frac{n}{3} \rfloor$ edges can cover all faces of a plane triangulation with n vertices. They also proved that the faces of every plane graph with n vertices can be guarded by $\lfloor \frac{2n}{5} \rfloor$ edges. See also [2] for

*Posgrado en Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México, D.F. México, j_cano@uxmcc2.iimas.unam.mx

†Department of Mathematics and Statistics, University of Calgary, Calgary, AB, cdtoth@ucalgary.ca

‡Instituto de Matemáticas, Universidad Nacional Autónoma de México, D.F. México, urrutia@matem.unam.mx

other variants of guarding polyhedral terrains in $\mathbb{R}^3$

For *orthogonal* polyhedra with m edges in $\mathbb{R}^3$, it was conjectured that $\frac{m}{12}$ edge guards are always sufficient [14]. For every $k \in \mathbb{N}$, there are *orthogonal* polyhedra P_k in $\mathbb{R}^3$ with $12(k+1)$ edges that require at least k edge guards [14]. Recently, Benbernou et al. [1] showed that $\frac{11m}{72}$ edges are always sufficient.

Benbernou et al. [1] also introduced a variant of the problem with *open* edge guards. An open edge e of P is visible from a point p if there is a point q in the relative interior of e such that p and q are visible in P. They showed that every orthogonal polyhedron of genus g with m edges can be guarded with $\frac{11m}{72} - \frac{g}{6} - 1$ open edge guards.

2 Edge covers in 3-connected graphs

An *edge cover* of a graph $G = (V, E)$ is a set of edges $E_1 \subseteq E$ such that every vertex $v \in V$ is incident to an edge in E_1. A minimum edge cover is the union of a maximum matching $M \subset E$ and one extra edge for each vertex not covered by M. Hence the size of a minimum edge cover is $|V| - |M|$.

Nishizeki and Baybars [2, 9] proved that the maximum matching in a 3-connected planar graph with n vertices has at least $(n+4)/3$ edges; and so every such graph has an edge cover of size at most $(2n-4)/3$. An edge cover of this size can be computed in $O(n)$ time [12]. If G is a maximal planar graph (a triangulation) with $n \geq 3$ vertices and $m = 3n-6$ edges, then G has an edge cover of size at most $\frac{2}{9}m$. However, we are interested in the minimum edge cover of an *arbitrary* 3-connected graph in terms of the number of *edges*, rather than the number of vertices of the graph.

We recall a few technical terms and the Edmonds-Gallai Structure Theorem for maximal matchings [8, 15]. Let $G = (V, E)$ be a simple graph. A matching $M \subset E$ is *perfect* if it covers all vertices of G; it is *near perfect* if it covers all but one vertex of G. According to the Edmonds-Gallai Structure Theorem, if $M \subset E$ is a maximum matching of G, then there is a vertex set $U \subseteq V$ (a Berge-Tutte witness set) with the following properties:

- M contains a perfect matching on every even component of $G[V \setminus U]$;
- M contains a near perfect matching on every odd component of $G[V \setminus U]$;
- M matches all vertices of U to vertices in distinct odd components of $G[V \setminus U]$.

A minimum edge cover of G can be obtained by augmenting the maximum matching M with one extra edge for each odd component of $G[V \setminus U]$ that is not fully covered by M. We are now in the position to prove the following lemma.

Lemma 1 *Every 3-connected graph with $n \geq 4$ vertices and m edges contains an edge cover of size at most $\lfloor (m+1)/3 \rfloor$. This bound is the best possible.*

Proof. Let $G = (V, E)$ be a 3-connected planar graph $|V| \geq 4$ vertices and $m = |E|$ edges. Let $M \subseteq E$ be a maximum matching of G. The Edmonds-Gallai Structure Theorem yields a Berge-Tutte witness set $U \subset V$.

If $U = \emptyset$, then $G[V \setminus U] = G$ has a unique connected component, in which M is a perfect or near perfect matching with at least $\lfloor |V|/2 \rfloor$ edges. In this case, G has an edge cover of size $\lceil |V|/2 \rceil$. Since G is 3-connected, the minimum vertex degree is 3, and $m \geq \lceil \frac{3}{2}|V| \rceil$. Then G has an edge cover of size at most $\lfloor (m+1)/3 \rfloor$.

Assume now that $U \neq \emptyset$. Denote the components of $G[V \setminus U]$ by $G_i = (V_i, E_i)$, for $i = 1, 2, \ldots, \ell$. Let $\overline{E}_i \subset E$ denote the set of all edges incident to vertices in V_i, that is, all edges in E_i and edges between U and V_i. The edge sets E_i, $i = 1, \ldots, \ell$, are pairwise disjoint. Since G is 3-connected, the minimum vertex degree is 3, and so the sum of degrees of the vertices in V_i is at least $3|V_i|$. Also, at least 3 edges in $\overline{E}_i$ are incident to some vertices in U. Hence $|\overline{E}_i| \geq \frac{3}{2}(|V_i| + 1)$.

If $|V_i|$ is even, then M contains a perfect matching on G_i, with $\frac{1}{2}|V_i|$ edges. Hence, the maximum matching M contains less than one third of the edges of $\overline{E}_i$.

If $|V_i|$ is odd, then M contains a near perfect matching on G_i, with $\frac{1}{2}(|V_i| - 1)$ edges. A minimum edge cover of G contains one more edge of $\overline{E}_i$ between U and V_i. Altogether, a minimum edge cover of G contains at most $\frac{1}{2}(|V_i| + 1)$ edges of $\overline{E}_i$. On the other hand, $|\overline{E}_i| \geq \frac{3}{2}(|V_i| + 1)$. Hence, a minimum edge cover contains at most a third of the edges of $\overline{E}_i$. Altogether, an upper bound $m/3$ follows in this case.

The bound $\lfloor (m+1)/3 \rfloor$ is the best possible. If $m \equiv 0$ or $m \equiv 1 \mod 3$, then the lower bound construction is a bipartite graph with vertex classes U and $V \setminus U$, where every vertex in $V \setminus U$ has degree 3. If $m \equiv 2 \mod 3$, then the lower bound construction is the 1-skeleton of a pyramid with a square base with 5 vertices and 8 edges (Figure 2). The base of the pyramid can be extended to a ladder for larger values of m. $\square$

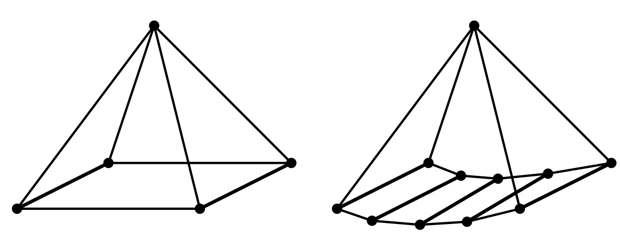


Figure 2: Lower bound constructions for $m \equiv 2 \mod 3$.

The 1-skeleton of a polyhedron in $\mathbb{R}^3$ is not necessarily connected (see Figure 1). However, each component of the 1-skeleton is 3-connected and has at least 4 vertices. For the edge cover of the 1-skeleton of a polyhedron, we derive the following corollary.

Corollary 2 *Let G be a graph such that every connected component of G is 3-connected and has at least 4 vertices. Then G has an edge cover with at most $\lfloor \frac{3m}{8} \rfloor$ edges. This bound is the best possible.*

Proof. Let $G_1, \ldots, G_k$ be the connected components of G, with $m_1, \ldots, m_k$ edges each. By Lemma 1, for each G_i we find an edge cover of size at most $\lfloor \frac{m_i+1}{3} \rfloor \leq \lfloor \frac{m_i}{3} \rfloor + 1$. Note that $\lfloor \frac{m_i+1}{3} \rfloor = \frac{m_i+1}{3}$ if $m_i \equiv 2 \mod 3$, and $\lfloor \frac{m_i+1}{3} \rfloor \leq \frac{m_i}{3}$ otherwise. Since $\sum_{i=1}^{k} m_i = m$, then

$$\sum_{i=1}^{k} \left\lfloor \frac{m_i+1}{3} \right\rfloor \leq \frac{m+k'}{3},$$

where k' is the number of components with $m_i \equiv 2 \mod 3$. Any such component has at least 8 edges, and so $k' \leq \lfloor \frac{m}{8} \rfloor$. It follows that

$$\frac{m+k'}{3} \leq \frac{m+\lfloor m/8 \rfloor}{3} \leq \frac{m+m/8}{3} = \frac{3m}{8},$$

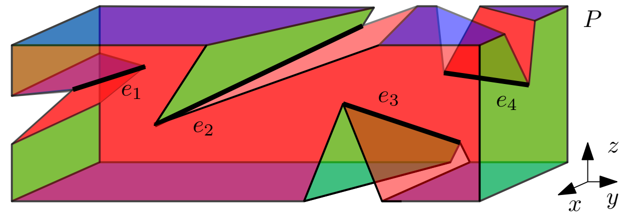
as required. This bound is tight if each component of G is a square pyramid as in Figure 2(left). $\square$

3 Four-coloring of edges in a polyhedron

Let P be a polyhedron with m edges (and arbitrary genus). Let $G = (V, E)$ denote the 1-skeleton of P. We may assume, by rotating P if necessary, that no edge in E is parallel to any coordinate plane. This ensures that the two endpoints of each edge $e \in E$ have distinct x- (resp., y- and z-) coordinates. We interpret above-below relation with respect to the z-axis (that is, a point a is above point b if a has a larger z-coordinate than b); and the left-right relation with respect to the y-axis. Recall that the boundary of P is a piecewise linear manifold, and so every edge $e \in E$ is incident to exactly two facets of P.

We distinguish between four types of edges in E as follows. For every edge $e \in E$, let H_e denote the plane spanned by e and a vertical line intersecting e. The plane H_e decomposes $\mathbb{R}^3$ into two halfspaces, lying on the left and the right of H_e. We say that e is a **left** edge if both facets incident to e lie in the left halfspace of H_e; edge e is a **right** edge if both facets incident to e lie in the right halfspace of H_e. The edge e is an **upper** edge if the two facets incident to e are in opposite halfspaces of H_e, and the interior of P lies below both facets. Edge e is a **lower** edge if the two facets incident to e are in opposite halfspaces of H_e, and the interior of P lies above both facets. See Figure 3 for examples.

We can now 4-color the edges of P such that the color classes correspond to the left, right, upper, and lower edges, respectively. We prove the following property of the 4-coloring.



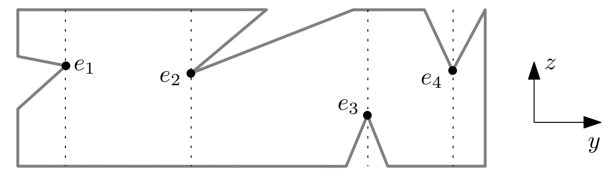


Figure 3: Top: An left edge e_1, a right edge e_2, a lower edge e_3, and an upper edge e_4 in a polyhedron P. Bottom: The cross-section of the polyhedron P with a plane parallel to the yz-plane, which is stabbed by edges $e_1, \ldots, e_4$. Dotted lines indicate the vertical lines passing through the the stabbing points of $e_1, \ldots, e_4$.

Lemma 3 *If a point $a \in P$ does not see any vertex of P, then a sees edges in at least two color classes.*

Proof. Let $a \in P$ be a point in the polyhedron P that does not see any vertex of P. Suppose that a sees edges of at most one color class. We distinguish four cases based on the color of the edges visible from a. By symmetry, it is enough to consider two out of four cases: left edges (the case of right edges is analogous), and upper edges (the case of lower edges is analogous).

Left edges. Suppose that every edge visible from a is a left edge. Consider the cross section of the polyhedron P with a plane H_a containing a and parallel to the yz-plane. Refer to Figure 4. The intersection $H_a \cap P$ may have several components, let P_a denote the component that contains a. Note that P_a is a 2-dimensional polygon, with possible holes. The vertices of P_a correspond to edges of P: each vertex of P_a is the intersection point of an edge of P with the plane H_a. Let V_a^* denote the set of reflex vertices of P_a that correspond to left edges of P. If $v \in V_a^*$, then the two edges of P_a incident to v lie on the left of v, and so the angle bisector of v is on the right side of a.

Decompose the polygon P_a as follows. Consider the vertices in V_a^* in an arbitrary order. From each vertex $v \in V_a^*$ successively shoot a ray along its angle bisector, and draw a segment along the ray from v to the first point where the ray hits the boundary of P_a or a previously drawn segment. If a ray hits a vertex, perturb the ray slightly so that it does not end at any vertex.

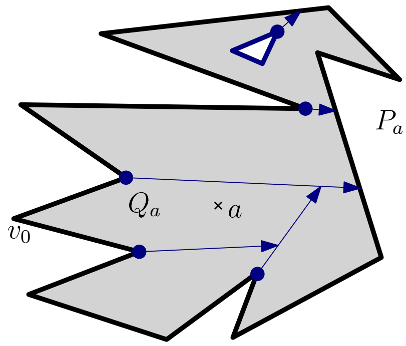


Figure 4: The polygon P_a is the cross-section of the polyhedron P with the plane H_a containing a and parallel to the xz-plane. The vertices in V_a^* are marked with large dots. P_a is decomposed into subpolygons by rays emitted by the vertices in V_a^*. The subpolygon Q_a contains a. Since Q_a is convex, a sees the leftmost vertex v_0 of Q_a.

The segments decompose P_a into subpolygons. Denote by $Q_a \subseteq P_a$ a subpolygon containing the point a, and let v_0 be the leftmost vertex of Q_a. Note that Q_a is a convex polygon, otherwise a sees a reflex vertex of Q_a which does not correspond to a left edge, since it would have no segment drawn along its bisector, contradicting the assumption that a only sees left edges. Since Q_a is convex, we have $av_0 \subset Q_a \subset P_a$, that is, v_0 is visible from a. Since all bisector rays are directed from left to right, v_0 has to be a vertex of the polygon P_a. Both edges of Q_a incident to v_0 are on the right side of v_0, as it is the leftmost vertex; and at least one of them has to be an edge of P_a, since every vertex of P_a emits at most one ray along its bisector. Therefore, v_0 does not correspond to a left edge of P. We have shown that a sees a non-left edge of P, contradicting our initial assumption.

Upper edges. Suppose that every edge visible from a is an upper edge. We decompose the polyhedron P into polyhedral cells such that each cell has exactly two nonvertical facets, which bound the cell from above and from below, respectively. We use (the first phase of) the standard vertical decomposition method [4, 13]. For every point p in every edge $e \in E$, erect a maximal vertical segment s_p such that $p \in s_p \subset P$. For an edge $e \in E$, the segments s_p, $p \in e$, form a vertical simple polygon A_e (which we call a *vertical wall*) whose upper and lower boundaries are contained in the boundary of P. The polygons A_e, $e \in E$, jointly decompose P into cells. Each cell has exactly two nonvertical facets, bounding the cell from above and below, respectively, and are contained in some facets of P; all other facets are contained in vertical walls corresponding to some edges of E. Due to the vertical walls A_e, $e \in E$, every cells has convex dihedral angles along the edges of the polyhedron P. A cell may still have a reflex dihedral angle at a *vertical* edge (e.g., consider the vertical decomposition of the polyhedron in Figure 1).

Denote by T_a a cell containing a. If point a sees some point p in a vertical wall A_e on the boundary of T_a, for some $e \in E$, then a sees the point $q \in e$ vertically above or below p. Recall that only upper edges of P are visible from a, hence every vertical wall A_e on the boundary of T_a visible from a corresponds to an upper edge $e \in E$.

We show that a sees some vertex of P. Assume first that T_a is nonconvex and so a sees some reflex edge e_r of T_a. Then e_r is a point p in a vertical edge of T_a, which lies on the boundary of two vertical walls, as noted above. Necessarily, a also sees a point vertically above p on the boundary of P, which is a vertex of P. Next assume that T_a is convex. Then every edge corresponding to a vertical wall on the boundary of T_a is incident to the top facet of T_a. Therefore, the top facet of T_a is bounded by edges of E, and hence it is a facet of P. Any vertex of the top facet of T_a is a vertex of P, and visible from a by convexity. We have shown in both cases that a sees some vertex of P. This contradicts our assumption that a does not see any vertex of P, and completes the proof. □

4 Obtaining the set of guards

The combination of the results in Sections 2 and 3 leads to the following bound on the minimum number of edge guards in a polyhedron.

Lemma 4 *Let P be a polyhedron with m edges in $\mathbb{R}^3$ (with arbitrary genus), and let E_1 be an edge cover of the 1-skeleton of P. Then P can be guarded by at most $(3m + |E_1|)/4$ edge guards.*

Proof. Four-color the edges of the 1-skeleton of P as described in Section 3. Place guards at all edges of E_1, and at the three smallest color classes of the remaining edges. Altogether, we use at most

$$|E_1| + \frac{3}{4}(m - |E_1|) = \frac{3m + |E_1|}{4}$$

edge guards. If a point $a \in P$ sees a vertex v, then it is guarded by an edge in E_1 that covers v. If a point $a \in P$ does not see any vertex of P, then it sees edges in at least two color classes by Lemma 3, and so it is guarded by an edge in one of the three smallest color classes. □

Finally, we prove our main results.

Theorem 5 *Every polyhedron in $\mathbb{R}^3$ with m edges (and arbitrary genus) can be guarded with at most $\frac{27}{32}m$ edge guards.*

Proof. Let P be a polyhedron with m edges in $\mathbb{R}^3$ with arbitrary genus. Let G be the 1-skeleton of P, and note that every connected component of G is 3-connected with at least 4 vertices. By Corollary 2, G has an edge cover E_1 of size $|E_1| \le \frac{3m}{8}$. By Lemma 4, P can be guarded by at most

$$\frac{3m + |E_1|}{4} \le \frac{3m + \frac{3m}{8}}{4} = \frac{27m}{32}$$

edges, as claimed. □

If the 1-skeleton of P is connected, we can establish a better upper bound.

Theorem 6 *Every polyhedron in $\mathbb{R}^3$ with m edges (and arbitrary genus) and a connected 1-skeleton can be guarded with at most $\frac{5}{6}m + \frac{1}{12}$ edge guards.*

Proof. Let P be a polyhedron with m edges in $\mathbb{R}^3$ with arbitrary genus. Let G be the 1-skeleton of P. By Lemma 1, G has an edge cover E_1 of size $|E_1| \le \frac{m+1}{3}$. By Lemma 4, P can be guarded by at most

$$\frac{3m + |E_1|}{4} \le \frac{3m + \frac{m+1}{3}}{4} = \frac{10m + 1}{12}$$

edges, as claimed. □

Using the same technique, one can also show that if the 1-skeleton of P is a triangulation with m edges, then it has an edge cover of size at most $\frac{2}{9}m$, and it can be guarded by at most $\frac{29m}{36}$ edge guards.

References

[1] N. M. Benbernou, E. D. Demaine, M. L. Demaine, A. Kurdia, J. O'Rourke, G. Toussaint, J. Urrutia, and G. Viglietta, Edge-guarding orthogonal polyhedra, in *23rd Canadian Conf. Comput. Geom.*, Toronto, 2011.

[2] P. Bose, D. Kirkpatrick, and Z. Li. Worst-case-optimal algorithms for guarding planar graphs and polyhedral surfaces, *Comput. Geom.* **26** (2003), 209–219.

[3] V. Chvátal, A combinatorial theorem in plane geometry, *J. Combin. Theory Se. B* **18** (1976), 39–41.

[4] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* **5** (1) (1990), 99–160.

[5] S. Devadoss and J. O'Rourke, Discrete and Comptuational Geometry, Princeton University Press, 2011.

[6] H. Everett and E. Rivera-Campo, Edge guarding polyhedral terrains, *Comput. Geom.* **7** (3) (1997), 201–203.

[7] J. Kahn, M. Klawe, and D. Kleitman, Traditional galleries require fewer watchmen, *SIAM J. Algebraic and Discrete Methods* **4** (1983), 194–206.

[8] L. Lovász and M. D. Plummer, Matching Theory, vol. 29 of *Ann. Discrete Math.*, North-Holland, Amsterdam, 1986.

[9] T. Nishizeki and I. Baybars, Lower bounds on the cardinalitiy of the maximum matchings of planar graphs, *Discrete Math.* **28** (1979), 255–267.

[10] J. O'Rourke, Galleries need fewer mobile guards: a variation on Chvátal's theorem, *Geometriae Dedicata* **14** (1983), 273–283.

[11] J. O'Rourke, Art Gallery Theorems and Algorithms, Oxford University Press, 1987.

[12] I. Rutter and A. Wolff, Computing large matchings fast, *ACM Trans. Algorithms* **7** (2010), article # 1.

[13] M. Sharir and P. K. Agarwal, Davenport-Schinzel Sequences and Their Geometric Applications, Cambridge University Press, 1995.

[14] J. Urrutia, Art gallery and illumination problems, in *Handbook of Computational Geometry*, pp. 973–1027, North-Holland, Amsterdam, 2000.

[15] D. B. West, A short proof of the Berge-Tutte Formula and the Gallai-Edmonds Structure Theorem, *European J. Combin.* **32** (2011), 674–676.