Tiling Polyhedra with Tetrahedra

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Abstract

When solving an algorithmic problem involving a polyhedron in $\mathbb{R}^3$, it is common to start by partitioning the given polyhedron into simplier ones. The most common process is called triangulation and it refers to partitioning a polyhedron into tetrahedra in a face-to-face manner. In this paper instead of triangulations we will consider tilings by tetrahedra. In a tiling the tetrahedra are not required to be attached to each other along common faces. We will construct several polyhedra which can not be triangulated but can be tiled by tetrahedra. We will also revisit a nontriangulatable polyhedron of Rambau and a give a new proof for his theorem. Finally we will identify new families of non-tilable, and thus non-triangulable polyhedra.

1 Introduction and Definitions

One of the fundamental approaches found in computational geometry is to break a region into smaller or simpler pieces. What is simple depends on the application. The process of partitioning a closed region into triangles has been abstracted to higher dimensions, yet still bears the name triangulation. One of the classical applications of triangulation is the art gallery theorem which states the fewest number of guards needed to guard a two dimensional polygonal region.

In this paper we will be concerned with the triangulation of polyhedra, in particular identifying polyhedra which cannot be triangulated. We will give five known examples of non-triangulable polyhedra and provide another example to justify a more general type of partitioning which we call tiling by tetrahedra. We will use the general partition to revisit the analogue of example 5, providing a shorter proof. In doing so we will show another family of polyhedra which cannot be tiled by tetrahedra and thus is non-triangulable. Finally we will introduce more families of non-tilable polyhedra and pose a generalization to this family.

Definition 1 *A* ***triangulation*** *of a polytope* $P \in \mathbb{R}^d$ *is a collection of d-simplices that satisfies the following two properties:*
1. The union of all these simplices equals P*. (Union Property)*
2. Any pair of these simplices intersect in a common face (possibly empty). (Intersection Property)

In this paper, we restrict ourselves to partitions where the vertices of each tetrahedron is a subset of the vertex set of P. For further information on triangulation, we suggest the texts [2], [3], and [5].

We wish to introduce the concept of *tiling by tetrahedra*, which weakens the intersection property of *triangulation*.

Definition 2 *A* ***tiling by tetrahedra*** *of a polyhedron* $\boldsymbol{P}$ *is a collection of tetrahedra, all of whose vertices are vertices of* $\boldsymbol{P}$*, that satisfies the following two properties:*
1. The union of all these tetrahedra equals $\boldsymbol{P}$*. (Union Property)*
2. The intersection of any two tetrahedra (possibly empty) is a subset of a plane. (Intersection Property)

Remark 1 *Figure 1 is an example of a tiling of the cube which is not a triangulation.*

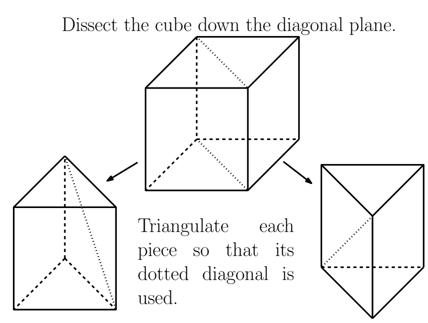

Figure 1: Tiling a cube

2 Non-Triangulable Polyhedra

It was first shown in 1911 by Lennes [4] that not all three-dimensional polyhedra are triangulable. We will provide eight other known examples of non-triangulable polyhedra in this section.

Example 1 (Schönhardt)
A frequently quoted and simple example was given by Schönhardt [10] in 1927. Schönhardt made a non-convex

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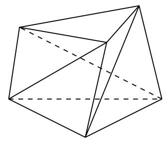

Figure 2: Schönhardt's twisted triangular prism

twisted triangular prism (Figure 2) by rotating the top face of a triangular prism so that a set of cyclic diagonals became edges with interior dihedral angles greater than 180^o.

Claim: Schönhardt's Twisted Triangular Prism cannot be triangulated.

Proof. Every diagonal of the polyhedron lies outside the polyhedron. Therefore any tetrahedron containing four vertices of the twisted triangular prism will contain at least one edge lying outside the polyhedron. □

Example 2 (Bagemihl)

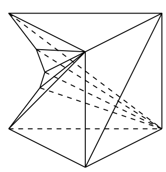

Figure 3: Bagemihl's generalization

In 1948, Bagemihl [1] modified Schönhardt's idea to construct a nonconvex polyhedron on $n \geq 6$ vertices by replacing one of the twisted vertical edges with a concave curve and placing $n-6$ vertices along the curve so that the interior dihedral angles of the edges to these vertices are greater than 180^o.

Claim: Bagemihl's Generalization cannot be triangulated.

Proof. If a triangulation exists, then the top triangular face must be a face of some tetrahedron. For every vertex v, not on the top face, there is a diagonal from v to some vertex on the top face which lies outside the polyhedron. Therefore there is no tetrahedron from the vertex set which has the top face as a boundary lying inside the polyhedron. □

Example 3 (Ruppert and Seidel)

Another method of creating non-triangulable polyhedra with large number of vertices was presented by Ruppert and Seidel [9]. They attached a copy of a non-triangulable polyhedron to another polyhedron. Figure 4 shows a polyhedron where a copy of Schönhardt's

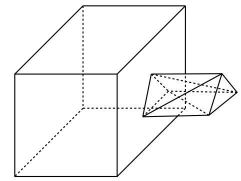

Figure 4: Attaching a niche to a cube

non-convex twisted triangular prism, called a niche, is attached to a face of a cube along a base of the twisted triangular prism.

Claim: If a niche is attached properly, the union of the polyhedron and the niche cannot be triangulated.

Proof. It can be arranged that the vertices of the Schöhardt prism which do not lie on the face of the cube do not see any vertex of the cube. Since each diagonal to the non-attached base of the triangular prism lies outside the polyhedron, then there must exist a tetrahedron contained inside the non-convex twisted triangular prism. We know from Example 1 this is not possible, so no set of tetrahedra triangulates the union. □

Example 4 (Thurston et al.)

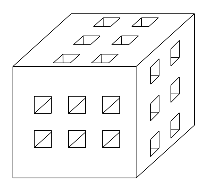

Figure 5: Thurston polyhedron

Figure 5 was attributed to Thurston by Paterson and Yao [7], where 18 non-intersecting square prisms, six from each pair of parallel faces, are removed from the cube. It is important to note that this polyhedron was independently discovered by several people including W. Kuperberg, Holden, and Seidel.

Claim: Thurston's polyhedron cannot be triangulated.

Proof. A point in a polyhedron "sees" another point in the polyhedron if the line segment between the two points is contained inside the polyhedron. We observe that each point of a tetrahedron can see each of the tetrahedron's vertices. If a polyhedron contains a point which does not see at least four non-coplanar vertices of the polyhedron, then it cannot be contained in a tetrahedron from the triangulation. In Thurston's polyhedron, the center of the cube does not see any vertex of the polyhedron, so it is obviously not in the interior of a tetrahedron of a triangulation. □

Example 5: (Rambau)

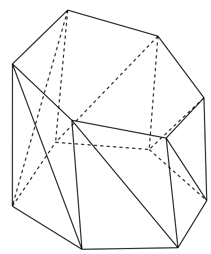

Figure 6: Twisted prism S_{C_6}

Rambau [8] provided another generalization of the Schönhardt twisted triangular prism. To construct the Nonconvex Twisted Prism we will first define a right prism over a convex polygon with n vertices, C_n. Label the vertices of C_n clockwise as $v_1, v_2, ..., v_n$. He defines the *right prism over* C_n as $P_{C_n} = \text{conv}\{(C_n \times \{0\}) \cup (C_n \times \{1\})\}$.

Now pick a point O in the interior of C_n and rotate C_n clockwise about O by ϵ, and label the vertices of $C_n(\epsilon)$, $v_1(\epsilon), v_2(\epsilon), ..., v_n(\epsilon)$, corresponding to the vertices of C_n.The **convex twisted prism over** C_n is $P_{C_n}(\epsilon) = \text{conv}\{(C_n \times \{0\}) \cup (C_n(\epsilon) \times \{1\})\}$.
The **non-convex twisted prism over** C_n (Figure 6) is $S_{C_n} = P_{C_n}(\epsilon)$ - conv$\{(v_i,0),(v_{i+1},0),(v_i(\epsilon),1),(v_{i+1}(\epsilon),1)\}$, for all $i \in (1, n)$ taken modulo n.

In [8] Rambau proves:

Theorem 1 *For all $n \geq 3$, no prism P_{C_n} admits a triangulation without new vertices that uses the cyclic diagonals $\{(v_i,0),\ (v_{i+1},1)\}$.*

Which implies

Corollary 2 *For all $n \geq 3$ and all sufficiently small $\epsilon > 0$, the non-convex twisted prism S_{C_n} cannot be triangulated without new vertices.*

The proof of Theorem 1 is too long to discuss here, but we will provide a shorter proof in the following section for Corollary 2.

3 Tilling by Tetrahedra

Notice that Rambau's results do not imply that S_{C_n} cannot be tiled with tetrahedra. Rambau uses Theorem 1 to conclude that no triangulation of S_{C_n} exist, but Figure 1 clearly shows that a tiling by tetrahedra exists for P_{C_4}, which is not a triangulation. Furthermore, this shows that there exists such a tiling which uses the cyclic diagonals of the cube. We prove that:

Theorem 3 *There exist a polyhedron which is not triangulable, but can be tiled by tetrahedra.*

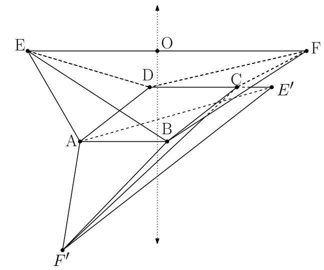

Figure 7: A non-triangulable polyhedron which can be tiled with tetrahedra

Proof. Example 6 will provide this result.

Example 6

Start with a horizontal unit square Q. Let A, B, C and D be the vertices of Q in counterclockwise order when we look down at the square from above. Choose the point O over Q at unit distance from its vertices. Next add to this arrangement a segment EF, whose midpoint is O, has length 4, and which is parallel to AB (assume E is closer to A than to B). Rotate this segment clockwise (i.e. opposite to the order of the vertices A, B, C and D) around the vertical line through O by a small angle ϵ. Let P be a non-convex polyhedron bounded by Q and by six triangles EAB, EBF, BFC, CDF, EFC, and EDA.

Finally let P' be the image of P under the reflection around the plane of Q followed by a $90°$ rotation around the vertical line containing O. Label the images of E and F as E' and F' respectively.

First notice that P is triangulable as it is the union of the tetrahedra $EABD, EBDF$ and $DBCF$. Since the same holds for P' we have that the union of P and P' can be tiled by tetrahedra.

Next we show that the union of P and P' is not triangulable. Since neither E nor F can see the vertices E' and F', we have that any triangulation of the union is the union of triangulations of P and P'. The polyhedron P was constructed so that the dihedral angles corresponding to the edges EB and FD are concave, therefore the diagonals AF and EC lie outside of P. It is easy to see that the triangles ABC and ACD cannot be faces of disjoint tetrahedra contained in P, thus diagonal BD must be an edge of at least one tetrahedron in any triangulation of P. A similar argument applied for P' gives that the diagonal AC is an edge of at least one tetrahedron in any triangulation of P'. Thus the union of P and P' is not triangulable. $\square$

Observation 1 *A non-triangulable polyhedron is tilable only if it contains at least four coplanar vertices where no three are incident with a common face. (We wish to thank one of the referees for this helpful observation)*

Since S_{C_n} does not contain 4 coplanar points, for sufficiently small ϵ, where no three are incident with a common face, Remark 1 implies that no tiling exists. However we wish to provide a shorter proof than that of Theorem 1 and provide a more general family of non-tilable polyhedra.

We will look at possible tetrahedra contained inside a polyhedron and determine if any two interior tetrahedra intersect. If two tetrahedra intersect in more than a plane, we can conclude that both tetrahedra cannot be in the tiling.

Lemma 4 *Let two tetrahedra T_O and T_B share an edge e and contain two coplanar faces t_O and t_B respectively on a plane P. If there exists a plane $Q \neq P$ containing e such that the fourth vertex O of T_O is in the open halfplane bounded by Q containing t_B, and such that the fourth vertex B of T_B is in the open halfplane bounded by Q containing t_O, then $T_O \cap T_B$ is a polyhedron, hence the interiors of T_O and T_B are not disjoint.*

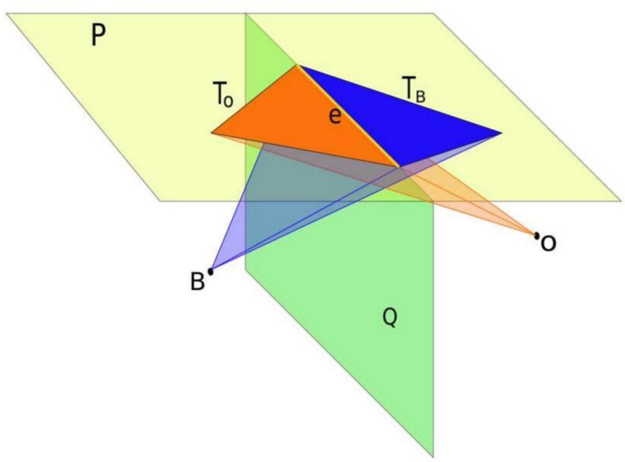

Figure 8: Edge sharing tetrahedra which cross

Lemma 4 can simply be proven by noticing that the interior dihedral angle of T_O, created by the faces t_O and $\{E, O\}$, and the interior dihedral angle of T_B, created by the faces t_B and $\{E, B\}$, sum to greater than 180^o.

Remark 2 *We will use Lemma 4 to say T_O and T_B cannot both be tetrahedra of a tiling by tetrahedra.*

Since each face of a tetrahedron $t \in T$ is a triangle, we say T induces a surface triangulation. Rambau used this observation in proving Corollary 2 by using Theorem 1. We will also use this observation to determine which tetrahedron contains a particular surface triangle as a face.

Definition 3 *An ear of a 2-dimensional triangulation of a polygon P is a triangle with exactly two of its edges belonging to P.*

Theorem 5 *(Meisters [6]) For $n > 3$, every triangulation of a polygon has at least 2 ears.*

It is common to view each triangulation as a tree by letting each triangle be represented by a dual vertex where two dual vertices are adjacent if the corresponding triangles share an edge. In this dual tree each ear is a leaf. We will borrow the terminology of pruning a leaf, to prune ears of a triangulation.

Definition 4 *An ear, E, is pruned by deleting the ear from the triangulation, leaving the edge which was not an edge of P as an edge of $P' = P - E$. In doing so, we delete a vertex of the polygon.*

Example 7:

Define an infinite set of polyhedra B_{C_n} (Figure 10) as follows:

Let the bottom base be a convex polygon on n vertices, C_n, with vertices labeled clockwise as $b_1, b_2, ..., b_n$. Define l_i to be the line containing edge $\overline{b_i b_{i+1}}$ (indices taken modulo n). Now we will call the closed area bounded by the lines l_i, l_{i-1}, and l_{i-2}, which contains $\overline{b_{i-1} b_i}$ but does not contain C_n, region R_i (Figure 9). (Region R_i may be infinite if l_i and l_{i-2} are parallel or intersect on the same side of l_{i-1} as the polygon.)

Now define the upper base as the convex polygon $U_n = \text{conv}\{u_i, u_2, ..., u_n\}$, where $u_i \in R_i$.

Let $B'_{C_n} = \text{conv}\{(C_n \times \{0\}) \cup (U_n \times \{1\})\}$, so that $B_{C_n} = B'_{C_n} - \text{conv}\{(b_i,0),(b_{i+1},0),(u_{i+1},1),(u_{i+2},1)\}$, for all $i \in \{1, 2, \ldots, n\}$ taken modulo n.

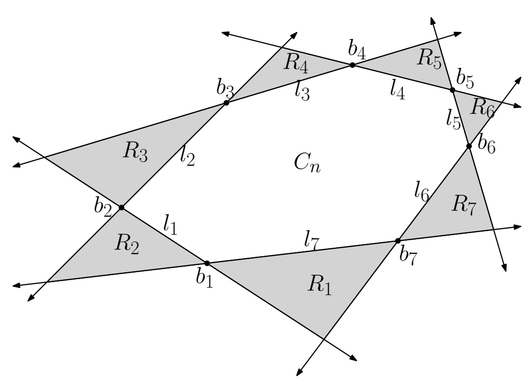

Figure 9: All regions R_i for C_7

Theorem 6 *The non-convex polyhedron B_{C_n} cannot be tiled with tetrahedra.*

Proof. Assume a set of simplices (tetrahedra) S tiles B_{C_n}. The tiling by S induces a triangulation of $(U_n \times$

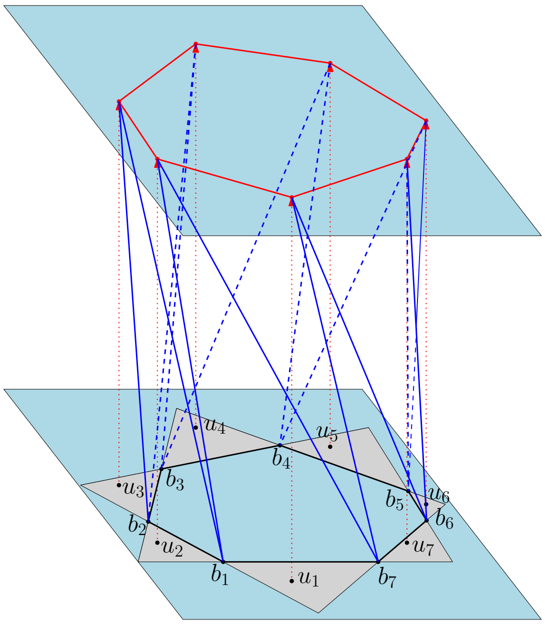

Figure 10: B_{C_7}

$\{1\}$), which we will call T. Now, for every $t \in T$ there exists exactly one $s \in S$ such that t is a face of s. Obviously, the fourth vertex of s must be a vertex of $(C_n \times \{0\})$.

Define a *sub-polygon* to be the convex hull of a subset of the vertices of a polygon. Let P be the set of sub-polygons of U_n such that every edge of a sub-polygon $p \in P$ is an edge of some $t \in T$.

Let e be an edge of p and let t be a triangle of T inside p, having e as an edge. We will say p is *separating* if every point b_i in the open halfplane bounded by the line containing e which does not contain p, is not in a tetrahedron of S with t.

Let $P' \subseteq P$ so that every $p' \in P'$ is separating. P' is not empty since U_n is a separating sub-polygon.

Let a minimal separating sub-polygon be the sub-polygon with the fewest vertices.

Let $m \in P'$ be a minimal separating sub-polygon with n vertices. If $n > 3$, then there is a $t \in T$ which is an ear of m. Let d be the edge of t which is not an edge of m. Observe that there exists a triangle $t' \in T$ which has d as an edge and is contained in m.

Remark 3 *The construction of U_n yields the property that the line containing the diagonal $\overline{u_i u_j}$ (for $i < j$) bounds two open halfplanes such that the halfplane containing the vertices u_k for $i < k < j$ also contains the vertices b_m for $i \leq m < j$ and no other vertices from the polygon C_n.*

Let Q be the plane through d, perpendicular to the plane containing U_n. Since m is separating, we can conclude by Lemma 4 that t' cannot be in a tetrahedron with any $(b_i, 0)$ where b_i is in the open halfplane, bounded by the line containing d, containing t. Therefore we can prune t so that $m - t$ is a separating sub-polygon with fewer vertices than m. Thus m is not a minimal separating sub-polygon. Therefore we can conclude that the minimal separating sub-polygon is a triangle.

So there is a $t = \{u_x, u_y, u_z\} \in T$ which is a minimal separating sub-polygon. Since t is separating, for every b_i outside of t, t is not in a tetrahedron with $(b_i, 0)$. By Remark 3, the only vertices which can exist inside t are b_x, b_y, or b_z, but the segments $\overline{(b_i,0)(u_i,1)}$ lie outside the polyhedron by the construction of B_{C_n}. Thus there exists a surface triangle which is not the face of a tetrahedron of S. Therefore there is no set of tetrahedra which tiles B_{C_n}. $\square$

A closer look at the proof yields that Remark 3 is the only observation necessary of U_n for the proof. Thus we will define a particular alteration, A_{C_n}, of a prism.

Let C_n be the same convex polygon defined in B_{C_n}. Let $A_n = \text{conv}\{a_1, a_2, ..., a_n\}$, where the line containing the diagonal $\overline{a_i a_j}$ (for $i < j$) bounds two open halfplanes such that the halfplane containing the vertices a_k for $i < k < j$ also contains the vertices b_m for $i \leq m < j$ from the polygon C_n. Let $A'_{C_n} = \text{conv}\{(C_n \times \{0\}) \cup (A_n \times \{1\})\}$. The *non-convex altered prism over C_n* is $A_{C_n} = A'_{C_n} - \text{conv}\{(b_i,0),(b_{i+1},0),(a_i,1),(a_{i+1},1)\}$, for all $i \in (1, n)$ taken modulo n.

Corollary 7 *For all $n \geq 3$ the non-convex altered prism A_{C_n} cannot be tiled by tetrahedra, hence it also cannot be triangulated.*

Remark 4 *Upon close inspection, it is easy to see that there is a convex polygon C_n where no rotational center yields the observations made in Remark 3. Such an example is provided on the coordinate plane in Figure 11.*

We note that, for small rotations, if the center of rotation lies on a point with a positive or 0 x-coordinate, then the diagonal $\overline{(-1,1)(-3,1-\epsilon)}$ will not satisfy Remark 3. Similarly, if the center of rotation lies on a point with a negative or 0 x-coordinate, then the diagonal $\overline{(1,-1)(3,-1+\epsilon)}$ will not satisfy Remark 3.

Theorem 8 *For all $n \geq 3$ and all sufficiently small $\epsilon > 0$, the non-convex twisted prism S_{C_n} cannot be tiled by tetrahedra without new vertices.*

Proof. It suffices to show that for any C_n, there exists a sufficiently small ϵ such that for any diagonal $\overline{(v_i,1)(v_j,1)}$ (for $i < j$) of $C_n(\epsilon)$ there is a plane Q containing the diagonal $\overline{(v_i,1)(v_j,1)}$ which bounds two open halfspaces such that the halfspace containing the

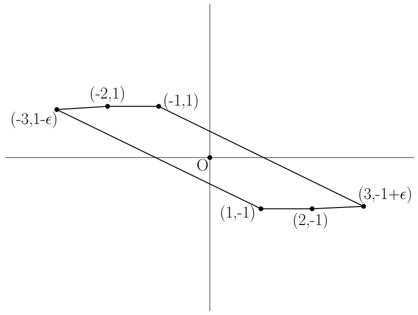

Figure 11: No rotational center

vertices $(v_k, 1)$ for $i < k < j$ also contains the vertices $(v_m, 0)$ for $i \leq m < j$ and no other vertices from the polygon $C_n \times \{0\}$. When constructing $C_n(\epsilon)$ we must consider the planes through each diagonal. Now, for any rotational center O, there is some angle of rotation α_{ij} where the diagonal $\overline{v_i(\alpha_{ij})v_j(\alpha_{ij})}$ lies on a line parallel to the diagonal $\overline{v_{i-1}v_{j-1}}$. Thus, for every S_{C_n} where $0 < \epsilon_{ij} < \alpha_{ij}$ there exists a plane Q satisfying the conditions of Lemma 4 for the diagonal $\overline{(v_i, 1)(v_j, 1)}$. It follows that if we let $\alpha = \min\{\alpha_{ij} | i, j \in (1, 2, 3, ..., n)\}$, then for every ϵ, $0 < \epsilon < \alpha$, S_{C_n} cannot be tiled by tetrahedra. $\square$

Example 8: (Nonconvex Twisted Dodecahedron)

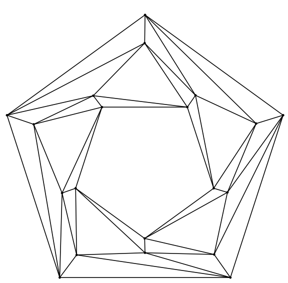

Figure 12: Planar representation of $DH(\epsilon)$

Let two parallel faces of a regular dodecahedron be the Bottom and Top faces. Let l be the line through the center of the two parallel faces. Rotate the Bottom face about l counterclockwise by an angle $\beta \leq \epsilon$, and the Top face about l clockwise by an angle $\tau \leq \epsilon$. The nonconvex twisted dodecahedron $DH(\epsilon)$ (Figure 12) is obtained by taking the convex hull of the 20 points and then removing the convex hull of each set of five points which was the face of the dodecahedron, with the exception of the Top and Bottom faces.

Theorem 9 *For sufficiently small ϵ the nonconvex twisted dodecahedron cannot be tiled by tetrahedra.*

A generalization of Lemma 4 and the argument used for Theorem 6 suffice to show Theorem 9 to be true. Furthermore we believe the previous known techniques would not be able to show this family of polyhedra is non-triangulable.

4 Open Problem

The result for $DH(\epsilon)$ motivates a generalization, as Schönhardt's twisted triangular prism motivated Rambau's generalization.

Notice that the position of the Top and Bottom faces of the regular dodecahedron is the same as the right pentagonal anti-prism. A n-gonal pentaprism PP_n is bounded by two congruent regular n-gonal bases in the same position as the right n-sided anti-prism and $2n$ pentagonal lateral faces, one adjacent to each edge of a base. If δ is the interior dihedral angle of the right n-sided anti-prism, then let the angle between a base and a lateral pentagon be $\delta < \alpha < 180$. A non-convex twisted n-gonal pentaprism $PP_n(\epsilon)$ is created by twisting the Top and Bottom faces of PP_n as in $DH(\epsilon)$.

Remark 5 *$DH = PP_5$ for $\epsilon = \arccos(\frac{-1}{\sqrt{5}})$.*

We leave the reader with this open problem. Is the non-convex twisted PP_n non-tilable by tetrahedra for all $n > 3$?

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