Visibility-Monotonic Polygon Deflation*

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Abstract

A *deflated* polygon is a polygon with no visibility crossings. We answer a question posed by Devadoss et al. (2012) by presenting a polygon that cannot be deformed via continuous visibility-decreasing motion into a deflated polygon. In order to demonstrate non-deflatability, we use a new combinatorial structure for polygons, the directed dual, which encodes the visibility properties of deflated polygons. We also show that any two deflated polygons with the same directed dual can be deformed, one into the other, through a visibility-preserving deformation.

1 Introduction

Much work has been done on visibilities of polygons [6, 8] as well as on their convexification, including work on convexification through continuous motions [4]. Devadoss et al. [5] combine these two areas in asking the following two questions: (1) Can every polygon be convexified through a deformation in which visibilities monotonically increase? (2) Can every polygon be deflated (i.e. lose all its visibility crossings) through a deformation in which visibilities monotonically decrease?

The first of these questions was answered in the affirmative at CCCG 2011 by Aichholzer et al. [2]. In this paper we resolve the second question in the negative. We also introduce a combinatorial structure, the directed dual, which captures the visibility properties of deflated polygons and we show that a deflated polygon may be monotonically deformed into any deflated polygon with the same directed dual.

2 Preliminaries

We begin by presenting some definitions. Here and throughout the paper, unless qualified otherwise, we take *polygon* to mean simple polygon on the plane.

A *triangulation*, T, of a polygon, P, with vertex set V is a partition of P into triangles with vertices in V. The *edges* of T are the edges of these triangles and we call such an edge a *polygon edge* if it belongs to the polygon or, else, a *diagonal*. A triangle of T with exactly one diagonal edge is an *ear* and the *helix* of an ear is its vertex not incident to any other triangle of T.

Let w and uv be a vertex and edge, respectively, of a polygon, P, such that u and v are seen in that order in a counter-clockwise walk along the boundary of P. Then uv is *facing* w if (u, v, w) is a left turn. Two vertices or a vertex and an edge of a polygon are *visible* or *see* each other if there exists a closed line segment contained inside the closed polygon joining them. If such a segment exists that intersects some other line segment then they are visible *through* the latter segment. We say that a polygon is in *general position* if the open line segment joining any of its visible pairs of vertices is contained in the open polygon.

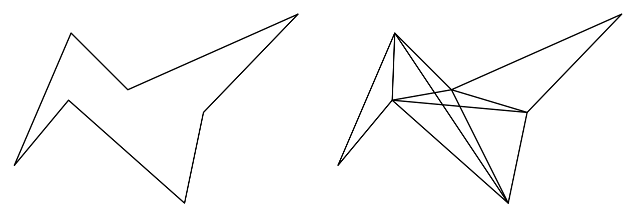

Figure 1: (a) A polygon and (b) its visibility graph.

The *visibility graph* of a polygon is the geometric graph on the plane with the same vertex set as the polygon and in which two vertices are connected by a straight open line segment if they are visible (e.g. see Figure 1).

2.1 Polygon Deflation

A *deformation* of a polygon, P, is a continuous, time-varying, simplicity-preserving transformation of P. Specifically, to each vertex, v, of P, a deformation assigns a continuous mapping $t \mapsto v^t$ from the closed interval $[0, 1] \subset \mathbb{R}$ to the plane such that $v^0 = v$. Additionally, for $t \in [0, 1]$, P^t is simple, where P^t is the polygon joining the images of t in these mappings as their respective vertices are joined in P.

A *monotonic deformation* of P is one in which no two vertices ever become visible, i.e., there do not exist u and v in the vertex set of P and $s, t \in [0, 1]$, with $s < t$, such that u^t and v^t are visible in P^t but u^s and v^s are not visible in P^s.

A polygon is *deflated* if its visibility graph has no edge intersections. Note that a deflated polygon is in general

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position and that its visibility graph is its unique triangulation. Because of this uniqueness and for convenience, we, at times, refer to a deflated polygon and its triangulation interchangeably. A *deflation* of a polygon, P, is a monotonic deformation $t \mapsto P^t$ of P such that P^1 is deflated. If such a deformation exists, then P is *deflatable*.

2.2 Dual Trees of Polygon Triangulations

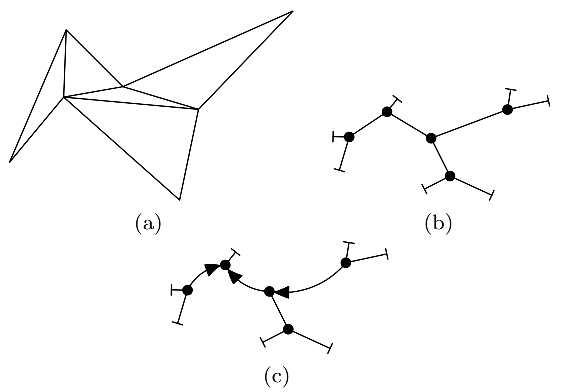


Figure 2: (a) A polygon triangulation, (b) its dual tree and (c) its directed dual. Triangle and terminal nodes are indicated with disks and tees, respectively.

The *dual tree*, D, of a polygon triangulation, T, is a plane tree with a *triangle node* for each triangle of T, a *terminal node* for each polygon edge of T and where two nodes are adjacent if their correspondents in T share a common edge. The dual tree preserves edge orderings of T in the following sense. If a triangle, a, of T has edges e, f and g in counter-clockwise order then the corresponding edges of its correspondent, a^D, in D are ordered e^D, f^D and g^D in counter-clockwise order (e.g. see Figure 2b).

Note that the terminal and triangle nodes of a dual tree have degrees one and three, respectively. We call the edges of terminal nodes *terminal edges*.

An ordered pair of adjacent triangles (a, b) of a polygon triangulation, T, is *right-reflex* if the quadrilateral union of a and b has a reflex vertex, v, situated on the right-hand side of a single segment path from a to b contained in the open quadrilateral. We call v the *reflex endpoint* of the edge shared by a and b (see Figure 3).

The *directed dual*, D, of a polygon triangulation, T, is a dual tree of T that is partially directed such that, for every right-reflex pair of adjacent triangles (a, b) in T, the edge joining the triangle nodes of a and b in D is directed $a \to b$ (e.g. see Figure 2c). Note that if P is deflated, then for every pair of adjacent triangles, (a, b), of T one of (a, b) or (b, a) is right-reflex and so every non-terminal edge in D is directed.

Throughout this paper, as above, we use superscripts

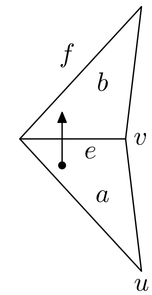


Figure 3: A pair of triangles, a and b, sharing an edge, e, such that their quadrilateral union has a reflex vertex and a single segment path from a to b contained in the open quadrilateral. The reflex endpoint, v, of e is to the right of the path and so the pair (a, b) is right-reflex.

to denote corresponding objects in associated structures. For example, if a is a triangle of the triangulation, T, of a polygon and b is a triangle node in the dual tree, D, of T then a^D and b^T denote the node corresponding to a in D and the triangle corresponding to b in T, respectively.

3 Directed Duals of Deflated Polygons

In this section, we derive some properties of deflated polygons and use them to relate the visibilities of deflated polygons to paths in their directed duals. We also show that two deflated polygons with the same directed dual can be monotonically deformed into one another. The proofs of Lemmas 1, 3 and 4 are not difficult and can be found in the full version of this paper [3].

Lemma 1 *Let P be a deflated polygon, let a be an ear of P and let P' be the polygon resulting from removing a from P. Then P' is deflated.*

Corollary 2 *If the union of a subset of the triangles of a deflated polygon triangulation is a polygon, then it is deflated.*

Lemma 3 *If u is a vertex opposite a closed edge, e, in a triangle of a deflated polygon triangulation, then u sees exactly one polygon edge through e.*

Let u be the vertex of a deflated polygon triangulation, T, and let e be an edge opposite u in a triangle of T. An *induced sequence* of u through e is the sequence of edges through which u sees a polygon edge, f, through e. This sequence is ordered by the proximity to u of their intersections with a closed line segment joining u and f that is interior to the open polygon everywhere but at its endpoints (e.g. see Figure 4b).

Lemma 4 *Suppose u is a vertex opposite a closed non-polygon edge, e, in a triangle, a, of a deflated polygon*

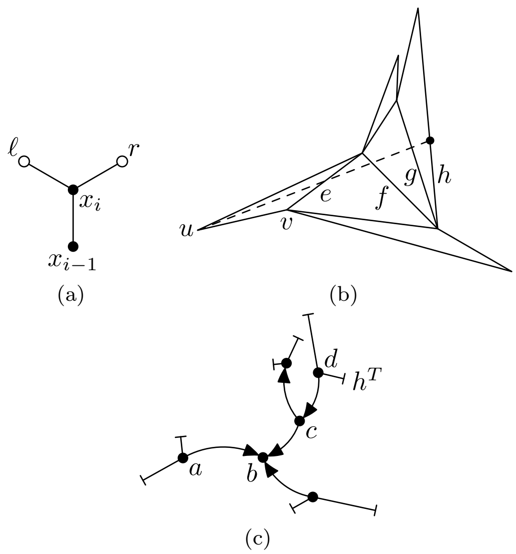


Figure 4: (a) A node x_i of a directed dual and its neighbours x_{i-1}, r and ℓ in an iteration of the construction of a visibility path, (b) a deflated polygon triangulation, T, wherein the induced sequence of the vertex u through the edge e is $(e,\ f,\ g,\ h)$ and (c) the directed dual, T, in which the visibility path of the directed dual starting with nodes (a, b) is $(a,\ b,\ c,\ d,\ h^T)$.

triangulation. Let v be the reflex endpoint of e and let f be the edge opposite v in the triangle sharing e with a (see Figure 3). Then u sees the same polygon edge through e as v sees through f.

Corollary 5 *If u, v, e and f are as in Lemma 4, then the induced sequence of u through e is equal to that of v through f prepended with e.*

3.1 Directed Duals and Visibility

A *visibility path*, $(x_1,\ x_2,\ \ldots,\ x_n)$, of the *directed dual*, D, of a deflated polygon is a sequence of nodes in D meeting the following conditions. x_1 is a triangle node adjacent to x_2 and, for $i \in \{2, \ldots, n\}$, if x_i is a terminal node, then it is x_n—the final node of the path. Otherwise, let the neighbours of x_i be x_{i-1}, r and ℓ in counter-clockwise order (see Figure 4a). Then

$$x_{i+1} = \begin{cases} r & \text{if edge } \{x_{i-1}, x_i\} \text{ is directed } x_{i-1} \leftarrow x_i \\ \ell & \text{if edge } \{x_{i-1}, x_i\} \text{ is directed } x_{i-1} \rightarrow x_i \end{cases}$$

(e.g. see Figure 4c).

Lemma 6 *Let (a, b, c) be a simple path in the directed dual, D, of a deflated polygon triangulation, T, where a and b are triangle nodes joined by the edge e. Let u be the vertex opposite e^T in a^T, let v be the reflex endpoint of e^T and let f be the edge opposite v in b^T (see Figure 4b). Then (a, b, c) is the substring of a visibility path if and only if f^D joins b and c in D.*

Proof. Suppose (a, b, c) is the substring of a visibility path and let x be the neighbour of b not a nor c and let x' be the edge of b^T not e^T nor f. We consider the case where the neighbours of b are a, x and c in counter-clockwise order—the argument is symmetric in the other case. Then (a, b) is right-reflex and so b^T has counter-clockwise edge ordering: e^T, x', f. Then, since edge orderings are preserved in the directed dual, f^D joins b and c as required. Reversing the argument gives the converse. □

Corollary 7 *Let D, T, a, b, e and u be as in Lemma 6. The induced sequence of u through e is equal to the sequence of correspondents in T of edges traversed by the visibility path starting with (a, b) in D. The final node of this visibility path corresponds to the edge u sees through e^T.*

Theorem 8 *A vertex, u, and edge, g, of a deflated polygon, P, are visible if and only if there is a visibility path in the directed dual, D, of the triangulation, T, of P starting on a triangle node corresponding to a triangle incident to u and ending on g^D.*

Proof. Assume u sees g. If g is an edge of a triangle, a, incident to u then (a^D, g^D) is the required visibility path. Otherwise u sees g through some edge, e, and the existence of the required visibility path follows from Corollary 7.

Assume, now, that the visibility path exists. If its triangle nodes all correspond to triangles incident to u then g is incident to one of these triangles and so visible to u. Otherwise, let e be the first edge the path traverses from a node, a, corresponding to a triangle incident to u to a node, b, corresponding to a triangle not incident to u.

Then, by Corollary 7, the induced sequence of u through e^T corresponds to a visibility path starting with (a, b) and this visibility path ends on a node corresponding to the edge u sees through e. Since two consecutive nodes of a visibility path determine all subsequent nodes, these visibility paths end on the same node, g^D, and so u sees g. □

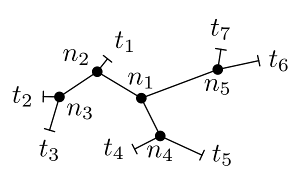


Figure 5: A plane tree with the following maximal outer paths: $(t_7,\ n_5,\ n_1,\ n_2,\ t_1)$, $(t_1,\ n_2,\ n_3,\ t_2)$, $(t_2,\ n_3,\ t_3)$, $(t_3,\ n_3,\ n_2,\ n_1,\ n_4,\ t_4)$, $(t_4,\ n_4,\ t_5)$, $(t_5,\ n_4,\ n_1,\ n_5,\ t_6)$, $(t_6,\ n_5,\ t_7)$.

An *outer path* of a plane tree, D, is the sequence of nodes visited in a counter-clockwise walk along its outer face in which no node is visited twice. An outer path is *maximal* if it is not a proper substring of any other outer path (e.g. see Figure 5). Note that an outer path, $(x_1, x_2, \ldots, x_n)$, of the directed dual of a polygon triangulation, T, corresponds to a triangle fan in T where the triangles have clockwise order $x_1^T, x_2^T, \ldots, x_n^T$ about their shared vertex.

Theorem 9 *A pair of vertices, u and v, of a deflated polygon P are visible if and only if, in the directed dual, D, of the triangulation, T, of P, their corresponding maximal outer paths share a node.*

Proof. The maximal outer paths of u and v share a node in D if and only if they are incident to a common triangle in T and, since P is deflated, this is the case if and only if u and v are visible. □

3.2 Directed Dual Equivalence

In this section, we show that if two deflated polygons have the same directed dual, then one can be monotonically deformed into the other. First, we fully characterize the directed duals of deflated polygons.

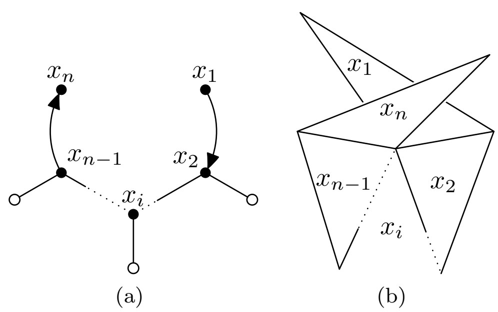


Figure 6: If (a) the tree with outer path $(x_1, x_2, \ldots, x_n)$ were a subtree of the directed dual of a polygon triangulation, T, then (b) the triangles corresponding to nodes x_1, x_2, x_{n-1} and x_n in T would overlap, contradicting the simplicity of the polygon.

Theorem 10 *A partially directed plane tree, D, in which every non-terminal node has degree three and where an edge is directed if and only if it joins two non-terminal nodes of degree three is the directed dual of a deflated polygon if and only if it does not contain an outer path, $(x_1, x_2, \ldots, x_n)$, with $n \geq 4$, such that the edges from x_1 and x_{n-1} are both forward directed (i.e. $x_1 \to x_2$ and $x_{n-1} \to x_n$).*

Henceforth, we call such a path an *illegal path.*

Proof. Suppose D contains an illegal path, $(x_1, x_2, \ldots, x_n)$. If D is the directed dual of a polygon triangulation, T, then x_1^T, x_2^T, x_{n-1}^T and x_n^T share a common vertex reflex in both quadrilaterals $x_1^T \cup x_2^T$ and $x_{n-1}^T \cup x_n^T$ (see

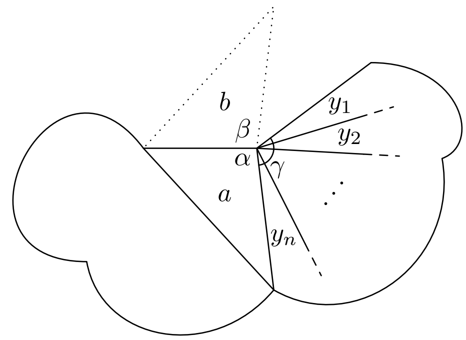


Figure 7: The inductive polygon in the proof of Theorem 10 or a polygon from the inductive deformation in the proof of Theorem 11.

Figure 6). This contradicts the disjointness of these quadrilaterals.

Suppose, now, that D has no illegal paths. We prove the converse with a construction of a polygon triangulation having D as its directed dual. Let b be a terminal node in the subtree of D induced by its non-terminal nodes. Then b has two terminal neighbours and one non-terminal neighbour, a. Let D' be the tree resulting from replacing a and its terminal neighbours with a single terminal node, x, connected to b with an undirected edge. By induction on the number of non-terminal nodes, there exists a deflated polygon triangulation, T, having D' as its directed dual.

Assume, without loss of generality, that the edge joining a and b is directed $a \to b$. Let u be the endpoint of x^T pointing in a clockwise direction in the boundary of T and let $(y_1, y_2, \ldots, y_n)$ be the outer path of D corresponding to the triangles other than b^T in T incident to u (see Figure 7). Note that $(y_i, y_{i+1}, \ldots, y_n, a, b)$ is an outer path of D and so, by hypothesis, for all $i \in \{1, 2, \ldots, n-1\}$, the edge joining y_i and y_{i+1} is directed $y_i \leftarrow y_{i+1}$.

Then, to show that a triangle may be appended to T to form the required triangulation, it suffices to show that the sum of the angles at u of the triangles y_1^T, y_2^T, $\ldots$, y_n^T is less than π, which, in turn, follows from the backward directedness of the edges of $(y_1, y_2, \ldots, y_n)$. □

Theorem 11 *If the deflated polygons P and P' have the same directed dual, D, then P can be monotonically deformed into P'.*

Proof. Let b be an ear of the triangulation, T, of P and let b' be the triangle corresponding to b^D in the triangulation, T', of P'. By induction on the number of triangles in T, there is a monotonic deformation $t \mapsto Q^t$ from $Q = P \setminus b$ to $Q' = P' \setminus b'$. Note that replacing b^D and its terminal nodes in D with a single terminal node gives the directed dual, D', of Q. Then, since Q

is deflated (Lemma 1) and $t \mapsto Q^t$ is monotonic, for all $t \in [0, 1]$, Q^t is deflated and has directed dual D'.

Let v be the helix of b, let a be the triangle sharing an edge, e, with b and let u be the reflex endpoint of e. We need to show that there is a continuous map $t \mapsto v^t$ that, combined with $t \mapsto Q$, gives a monotonic deformation of a polygon with directed dual D. For $t \in [0, 1]$, let α^t be the angle of a^t at u^t in Q^t and let γ^t be the sum of the angles at u^t of the triangles, y_1^t, y_2^t, …, y_n^t, other than a^t of the triangulation of Q^t incident to u^t (see Figure 7).

Then, since v may be brought arbitrarily close to u in a monotonic deformation of P, it suffices to show that there is a continuous map $t \mapsto \beta^t$ specifying an angle for b^t at u^t such that, for all $t \in [0, 1]$, $0 < \beta^t < \pi$, $\alpha^t + \beta^t > \pi$ and $\alpha^t + \beta^t + \gamma^t < 2\pi$. The latter two conditions are equivalent to

$$\pi - \alpha^t < \beta^t < (\pi - \alpha^t) + (\pi - \gamma^t) .$$

It follows from Theorem 10 that the outer path ($y_1^{D'}$, $y_2^{D'}$, …, $y_n^{D'}$) is left-directed and so that $\gamma^t < \pi$. Then $\beta^t = \pi - (\alpha^t + \gamma^t)/2$ satisfies all required conditions.

Now, let $t \mapsto R^t$ be the monotonic deformation from a polygon with directed dual D combining $t \mapsto Q^t$ and the map $t \mapsto v^t$ defined by a fixed distance between u^t and v^t of $r \in \mathbb{R}_{>0}$ and an angle for b^t at u^t of β^t.

Prepending $t \mapsto R^t$ with a deformation of P in which v is brought to the distance r from u and then rotated about u to an angle of β^0; then appending a deformation comprising similar motions ending at P'; and, finally, scaling in time gives a continuous map, $t \mapsto P^t$, with $P^0 = P$ and $P^1 = P'$. Since, for all $t \in [0, 1]$, Q^t is simple, a small enough r can be chosen such that $t \mapsto P^t$ is simplicity-preserving. Then, by the properties of $t \mapsto \beta^t$, $t \mapsto P^t$ is the required monotonic deformation. □

4 Deflatability of Polygons

In this section, we show how deflatable polygons may be related combinatorially to their deflation targets and use this result to present a polygon that cannot be deflated. We also show that vertex-vertex visibilities do not determine deflatability. These results depend on the following Lemma.

Lemma 12 *Let $t \mapsto P^t$ be a monotonic deformation of a polygon, P, in general position. Then a vertex and an edge are visible in P^1 only if they are visible in P.*

The proof, which is available in the full version of this paper [3], uses analytic arguments similar to those used by Ábrego et al. [1].

A *compatible directed dual* of a polygon, P, in general position is the directed dual of a deflated polygon, P', such that, under an order- and chirality-preserving bijection between the vertices of P and P', a vertex-edge or vertex-vertex pair are visible in P' only if their correspondents are visible in P. By *chirality*-preserving bijection, we mean one under which a counter-clockwise walk on the boundary of P corresponds to a counter-clockwise walk on the boundary of P'.

Theorem 13 *A polygon, P, in general position with no compatible directed dual is not deflatable.*

Proof. It follows from Lemma 12 that if P is monotonically deformable to a deflated polygon P', then the directed dual of P' is compatible with P. □

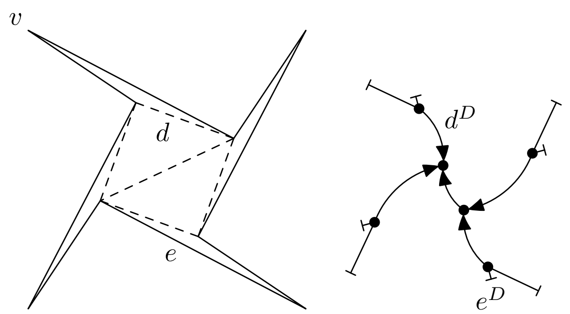
Lemma 14 *Suppose a polygon, P, in general position has a compatible directed dual, D. Let P' be the deflated polygon with directed dual D whose vertex-vertex and vertex-edge visibilities are a subset of those of P under an order- and chirality-preserving bijection. Then the unique triangulation, T', of P' is a triangulation, T, of P under the bijection and D can be constructed by directing the undirected non-terminal edges of the directed dual of T.*

Proof. Note that T' is the visibility graph of P'. Then, since P is in general position and has the same vertex count as P', it follows from the vertex-vertex visibility subset property of P' that T' triangulates P under the bijection.

It remains to show that, for every non-terminal edge of the directed dual of T, either the edge is undirected or it is directed as in D or, equivalently, that for every pair of adjacent triangles, a and b, in T corresponding to the triangles a' and b' in T', if (a, b) is right-reflex then so is (a', b'). Suppose, instead, that (b', a') is right-reflex. Let e' be the edge shared by a' and b', let u' be the vertex of a' opposite e' and let f' be the edge of b' opposite the reflex endpoint of e'. Then, by Lemma 4, u' sees an edge through f' but the corresponding visibility is not present in P, contradicting the vertex-edge visibility subset property of P'. □

Theorem 15 *There exists a polygon that cannot be deflated.*

Proof. We show that the general position polygon, P, in Figure 8a has no compatible directed dual and so, by Lemma 13, is not deflatable. Assume that the directed dual, D, of a deflated polygon, P', is compatible with P. Then, by Lemma 14, D can be constructed by directing the undirected non-terminal edges of the directed dual of some triangulation of P. Up to symmetry, P has a single triangulation, its directed dual has a single undirected non-terminal edge and there is a single way to direct this edge. Then we may assume, without loss of generality, that D is the tree shown in Figure 8b and, by



(a) (b)

Figure 8: (a) A non-deflatable polygon, P, with its only triangulation, up to symmetry, indicated with dashed lines and (b) its only candidate for a compatible directed dual, D, up to symmetry.

Theorem 8, the correspondents of the vertex v and edge e in P' are visible. This contradicts the compatibility of D. □

Note that, although the non-deflatability of P can be shown using *ad hoc* arguments, the combinatorial technique used here can be applied to other polygons. See the full version of this paper [3] for examples.

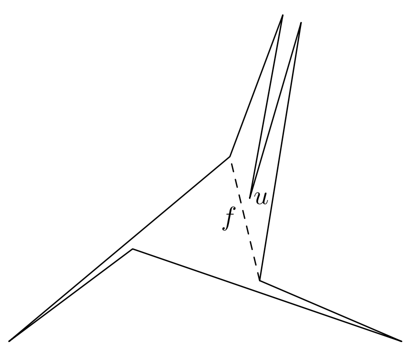


Figure 9: A deflatable polygon with the same vertex-vertex visibilities as the non-deflatable polygon shown in Figure 8a.

Theorem 16 *The vertex-vertex visibilities of a polygon do not determine its deflatability.*

Proof. The polygon in Figure 9 has the same vertex-vertex visibilities as the non-deflatable polygon in Figure 8a and yet can be deflated by moving the vertex u through the diagonal f. □

5 Summary and Conclusion

We presented the directed dual and showed that it captures the visibility properties of deflated polygons. We then showed that two deflated polygons with the same directed dual can be monotonically deformed into one another. Next, we showed that directed duals can be used to reason combinatorially, via directed dual compatibility, about the deflatability of polygons. Finally, we presented a polygon that cannot be deflated and showed that the vertex-vertex visibilities of a polygon do not determine its deflatability.

A full characterization of deflatable polygons still remains to be found. If the converse of Theorem 13 is true, then the existence of a compatible directed dual gives such a characterization. We conjecture the following weaker statement.

Conjecture 1 *The vertex-edge visibilities of a polygon in general position determine its deflatability.*

We conclude, however, by noting that, in light of Mnev's Universality Theorem [7], it is unknown if even the order type of a polygon's vertex set determines its deflatability.

6 Acknowledgements

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